Chapter 1 – Probability

Section 1.1 – What It Means

Suppose you roll a single, standard six sided die. How likely are you to get a six?

At first glance, that may seem like a pretty straightforward question. The die has six sides and only one of them has a six so the likelihood of getting that one value is going to be relatively small. But how small is "relatively small"? As a mathematician, that's a little too vague for me. I want a way to come up with a numeric answer. That's what probability is all about. Looking at questions like, "How likely are you to get a six?" and calculating an answer.

Before we go any further, we need some basic definitions. The table below has the definitions on the left and how they relate to our example on the left.

An **experiment** is an operation that has a well-defined set of results.	In our example, a single roll of the die is an experiment.
The **outcomes** are all the possible results of an experiment.	When you roll a die, there are six possible outcomes: 1, 2, 3, 4, 5 and 6.
An **event** is a collection of outcomes.	Rolling a 6 would be the event. Other possible events would be "rolling a 4" and "rolling a 3 or a 5".
A **trial** is a single run of the experiment.	Rolling the die would be a single trial.

The "well-defined" part of the definition may be a phrase you've never seen before. What it means is that there can't be any ambiguity in the outcomes. For example, if I asked you to determine the probability that a customer at an electronics store purchases a large television, the outcomes would *not* be well-defined. The problem is with the word "large". What I consider a large television might not be what you consider a large television. That vagueness is what makes the experiment not well-defined. We could fix it by specifying, for example, customers who purchased televisions greater than 17". There's no way for two people to have a different definition of 17" like there was with "large".

Suppose we took an experiment where we take a coin and flip it twice. There are several basic questions we can ask about this experiment that relate to our definitions.

Is this experiment well defined?	When you flip a coin, the result is either heads or tails. There's no possible ambiguity in the results so the experiment is well defined.
What are the possible outcomes?	For convenience, let H represent getting heads and let T represent getting tails. Then one outcome would be HH, i.e. getting heads on the first and second flips, another outcome would be HT, i.e. heads on the

	first flip and tails on the second. The only two other outcomes would be TH and TT.
What would a trial be?	A trial is a single attempt at the experiment. In this case, that would be flipping the coin twice.

The Basic Calculation

Now that we've got a basic understanding of what we're talking about, we're ready for the central formula in probability:

$$P(\text{an event}) = \frac{\text{number of outcomes that represent the event}}{\text{total number of outcomes}}$$

The numerator of that may be a little confusing so let's try looking at some examples.

Example 1 – Calculating Probabilities

Calculate the probability of rolling a 3 on a die.

First, the event that we're looking for is $P(\text{rolling a 3})$.

To use the formula to calculate this, we need to find two values: the total number of outcomes and the number of outcomes that include the event. The possible outcomes are 1, 2, 3, 4, 5 and 6. Since there are six total possible outcomes that makes our formula:

$$P(\text{rolling a 3}) = \frac{\text{number of outcomes that represent the event}}{6}$$

To find the numerator, we need to count the number of outcomes that represent a "success", i.e. getting the result that we want. Since there's only one roll that gives us a 3, the "number of outcomes that represent the event" is going to be 1. That makes our probability

$$P(\text{rolling a 3}) = \frac{1}{6}$$

Example 2 – Calculating Probabilities

Calculate the probability of rolling less than a 3 on a die.

Now the event that we're looking for is $P(\text{rolling less than 3})$. The total number of outcomes is still 6 just like it was in the previous example.

For the numerator, there are two rolls, 1 and 2, that meet the requirement that the result is less than 3 so our numerator is going to be 2. That makes the result:

$$P(\text{rolling less than a 3}) = \frac{2}{6} = \frac{1}{3}$$

There's a close relationship between probability, ratios and percentages that you can see in those two examples. In Example 2, for example, notice that 1/3 of the sides on the die meet the "less than 3" requirement and that ratio corresponds to the probability. That's not a coincidence. Another way to think of the probability that an event occurs is the fraction or ratio that a "success" represents.

You can also think of probabilities as being related to percentages but there's a little danger here. For Example 1, it would be reasonable to conclude from our calculations that a 3 is going to occur 16.7% of the time[1]. I would not want to say something like, "The probability is 16.7%." Keep in mind that probabilities are always going to be given as numbers, i.e. decimals or fractions, which aren't the same as probabilities. You can convert back and forth between decimals and percentages but it's incorrect to say that they're the same thing.

Some Basic Rules

Dice and cards are standard tools for probability examples that we'll be using throughout this section. There are a few rules you should keep in mind:

1. Unless you're told otherwise, assume that a die has six sides.
2. A deck of cards has 52 cards in it. We never count the jokers.
3. Face cards are kings, queens and jacks. Aces are never included in this group.

Example 3 – Calculating Probabilities

Calculate the probability of drawing a queen from

Example 4 – Calculating Probabilities

Calculate the probability of drawing a spade from a

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[1] 16.7% is 1/6 converted to a percentage: 1/6 = .1677… which is equivalent to 16.7%.

deck of cards.

First, the event that we're looking for is *P*(drawing a queen).

There are a total of 52 cards in the deck so the total number of possible outcomes will be 52. There are only 4 queens so the number of successful outcomes has to be 4. That makes the probability:

$$P(\text{drawing a queen}) = \frac{4}{52} = \frac{1}{13}$$

deck of cards.

Our event here will be *P*(drawing a spade). The total number of outcomes is 52, just like in Example 3. Since there are 13 spades in the deck, the number of successful outcomes is going to be 13. That makes our result:

$$P(\text{drawing a spade}) = \frac{13}{52} = \frac{1}{4}$$

Again this should make intuitive sense since the spades represent 1/4 of the entire deck.

Types of Probability

Probability comes in two varieties. The examples we've looked at so far are all what's referred to as "**theoretical probability**" or "**classical probability**". To do those calculations, you think about how many ways an event can occur and how many of those are successful then you substitute the results into the formula. The whole process is based on calculations and counting that can be done on a piece of paper.

Another kind of probability that's more closely tied to the real world is called "**empirical probability**". In this kind of calculation, you actually go out into the world, collect data and use the results in the formula.

Example 5 – Calculating Empirical Probabilities

Suppose you sat in front of the Magic Castle in Disney World and counted the number of people who walk by. After a half hour you had counted 25 couples, 12 single people and 72 families. Based on these results calculate the probability that the next person/group to walk by will be a family.

This is an example of empirical probability because our results are going to be based on the results of somebody's research rather than pure counting. Don't get distracted by this distinction. The process for calculating the result is exactly the same.

We want *P*(the next group is a family). The total number of outcomes from your research will be

$$25 + 12 + 72 = 109$$

Of those 109 results, 72 of them represent families so our probability will be

$$P(\text{a family}) = \frac{\text{number of families}}{\text{total number of groups}} = \frac{72}{109}$$

Example 6 – Calculating Empirical Probabilities

A researcher pulls 35 trout from a lake. 18 of the trout are female. Find the probability that the next trout will be male.

There's a little trick to this one. We're given the number of females but asked for the number of males. Presumably, all of the 35 trout who weren't female will be male so we can calculate the number of males by taking (males) = (total) – (females) = 35 – 18 = 17. Since the total number of fish in the sample was 35, that makes our probability

$$P(\text{male}) = \frac{\text{number of males}}{\text{total number of fish}} = \frac{17}{35}$$

Some Basic Rules

There are a few rules that you should be aware of before we go any further. They'll help you to check your answers and to make some calculations a little simpler.

Two Rules

1. For any event $0 \le P(x) \le 1$.
2. **The Compliment Rule:** *P*(an event happens) + *P*(the event doesn't happen) = 1.

A Note about Fractions

I know that nobody likes to work with

The first rule puts bounds on the probability of an event, i.e. the probability has to be between 0 and 1. To understand why this is true, it may help to think of probabilities in terms of percentages. There can't be more than a 100% chance that something happens and there can't be less than a 0% chance of something happening. As we get into more complex examples, this rule will give you a good way to do a quick reality check on your answers. If you go through a series of calculations and you end with a probability greater than one or less than zero then you'll know you made an error somewhere.

Two events that are opposites of each other, for example drawing a queen and not drawing a queen or rolling a three and not rolling a three, are called **complementary events** or **complementary results.** The second rule says that if you add the probabilities of complementary results, the result has to be 1. This also makes sense if you think of it in terms of percentages. If you draw a card from a deck, you'll get either a queen or you won't get a queen 100% of the time.

fractions. That's unfortunate since, as you probably noticed, the answers to probability questions tend to come out that way. As a general rule, fractions are going to be preferable to decimals since they're more exact so I'm going to leave all of my answers in that form. If you prefer to use decimals, I understand, just keep in mind that it's incorrect to say things like 1/3 = .333. The correct result would be 1/3 = .333…, i.e. a repeating decimal rather than a rounded one.

Example 7 – Finding Complimentary Probabilities

Find the probability of drawing a spade from a deck of cards. Find the probability of drawing a club, heart or diamond from a deck of cards.

To find *P*(drawing a spade) we would use

$$P(\text{spade}) = \frac{\text{number of spades}}{\text{number of cards}} = \frac{13}{52} = \frac{1}{4}$$

The second probability, drawing a club, heart or diamond, is the same as *not* drawing a spade so that probability would be:

$$P(\text{diamond, heart, club}) + P(\text{spade}) = 1$$

$$P(\text{diamond, heart, club}) + \frac{1}{4} = 1$$

$$P(\text{diamond, heart, club}) = 1 - \frac{1}{4} = \frac{3}{4}$$

Example 8 – Finding Complimentary Probabilities

A researcher pulls 35 trout from a lake. 18 of the trout are female. Find the probability that the next trout will be female.

In Example 6, we calculated the probability of getting a male was $\frac{17}{35}$. Applying our complementary events formula gives us:

$$P(\text{male}) + P(\text{female}) = 1$$

$$\frac{17}{35} + P(\text{female}) = 1$$

$$P(\text{female}) = 1 - \frac{17}{35} = \frac{18}{35}$$

Which is the same number we would have gotten if we had taken the number of females and divided it by the total number of fish in the sample.

What Probabilities Mean and What They Don't

Suppose you flip a regular, two-sided coin. The probability of getting tails is going to be 1/2, i.e.

$$\frac{(\text{one side of the coin has tails})}{(\text{two possible outcomes})}$$

Let's think for a minute about what that means. Suppose you flip the coin and get heads five times in a row. What's the probability of getting tails on the next flip?

Some people hesitate over this question. Since you got heads five times in a row, doesn't that mean you're "due" to get tails? The answer to that question is, no. The probability of getting tails on the next flip is 1/2, just like it is on the first flip. An important thing to remember about probability is that every event, i.e. every trial of an experiment, is independent of the ones that came before it. Another way to think of it is that probability doesn't have a memory. Past events are in the past and have no effect on the next trial.

So what does the probability tell us about how the coin behaves? Intuitively, it tells us that if you keep flipping a coin over and over again, half of the results are going to be heads and the other half tails. It doesn't tell you anything about an individual flip. For example, just because you got heads on the first toss, that doesn't mean you have to get tails on the next one to keep the ratio even. It's possible that you'll have "runs" of a single result, e.g. three heads in a row or six tails in a row. (We'll talk about how to calculate the probability of runs like that later in this chapter.) What the probability does tell you is how the results are going to even out in the long run. This idea is defined in what's called The Law of Large Numbers.

The Law of Large Numbers	If you do an experiment repeatedly, the empirical probability will get closer to the theoretical probability.

Another way to think of that is: The larger your sample size, the more accurate your result will be.

Exercises

Determine if each of the following experiments is well-defined. If it isn't, suggest a way to change it so that it is. For the well-defined experiments, list the possible outcomes.

1. You flip a coin ten times and use the results to calculate the probability of getting heads.
2. You survey people leaving an electronics store and ask them their opinion of the store's customer service.
3. A researcher measures the height of the high tide mark at a beach every day for a year.
4. Satellite data is used to calculate the average temperature at several points on the east coast.
5. A researcher wants to compare the time it takes white mice to complete a maze versus the time it takes gray mice.
6. Calculating the probability of drawing a red queen from a deck of cards.
7. Calculating the probability of drawing a card with a small number from a deck of cards.
8. Comparing the number of families to the number of non-families that enter a retail store.

Determine if the following situations are examples of empirical probability or theoretical probability.

9. Calculating the probability of drawing an ace from a deck of cards.
10. Flip a coin 100 times and calculate the probability of getting heads based on the results.
11. Calculate the probability that a randomly selected U.S. citizen is Hispanic based on census data.
12. Determine the probability of drawing a green marble from a bag by taking the number of green marbles in the bag and dividing it by the total number of marbles.
13. Calculate the probability of rolling a seven or greater on two dice.
14. Find the probability of selecting a given kind of fish from a river based on the current day's catch.

Calculate the following probabilities.

15. The probability of drawing a black queen from a deck of cards.
16. The population of a small town is 3,504 Republican, 2,512 Democrat and 378 independent. Find the probability that a randomly selected resident of the town will be a registered independent. (Assume that everyone in the two is registered and that Republican, Democrat and independent are the only options.)
17. The probability of drawing a green marble from a bag that has 3 red, 4 green and 5 blue marbles. The probability of drawing a red or blue marble, i.e. the probability of *not* drawing a green marble.
18. The probability of rolling a number greater than or equal to 4 on a single die.
19. A researcher collects clams from a beach every morning. After three days of collecting, he has 75 clams of Type A and 36 clams of Type B. What's the probability that the next clam he finds will be of Type A?
20. The probability of drawing a red card from a deck. The probability of drawing a black card from a deck.
21. Over the course of a week a veterinarian treated 35 cats, 22 dogs, 5 hamsters and a bird. Find the probability that the next pet brought into the clinic is a dog.
22. According to the Census Bureau, in 2000 there were approximately 140 million females and 134 million males in the United States. (*Population Profile of the United States: 2000 (Internet Release)*) What's the probability that the next person surveyed will be female?

Research Questions

23. Calculate the probability of getting heads when you flip a coin. Now take a coin, flip it 25 times and record the results. Based on your experiment, what's the empirical probability of getting heads? Flip the coin 25 more times and calculate the empirical probability based on all 50 flips.
24. When I was a teenager, Pepsi ran a series of commercials featuring what they called the "Pepsi Challenge" where they gave people unlabeled glasses of Coke and Pepsi and asked them if they could tell the difference. Try repeating this experiment: Give people that you know a glass of Coke and a glass of Pepsi and track the number who could tell the difference and the number who couldn't. Based on your research, what's the probability that someone can distinguish between the two drinks?
25. Ask your friends and relatives if they prefer sweet or salty foods for a snack. Use the data to calculate the probability that a randomly selected person will like each of the two types.

Section 1.2 – Sample Spaces, Tree Diagrams and Permutations

> Suppose you roll two, standard six sided dice. How likely are you to get a seven?

The first step in solving this is going to be finding the value of the denominator, i.e. counting the total number of possible results from rolling the two dice. For something relatively simple like this, you might be tempted just to write down all the combinations then count the results. That works well enough for things like how many fingers am I'm holding up or how many books are on a bookshelf but suppose I threw something more complicated like how many sandwiches can you make if you have a choice of three kinds of bread, six kinds of meat and four kinds of cheese? It's just not going to be practical to try to write down every possible combination of bread, meat and cheese, so we're going to need some methods that are a little more sophisticated than making a list.

The Basic Calculation

Before we begin you need to understand what it means to "decompose" a problem. In mathematics, it means to break a big problem down into smaller ones. In our specific situation, it's going to mean breaking a big choice down into smaller choices. Going back to our sandwich example, rather than thinking of the problem as one big one, i.e. picking a sandwich, we're going to think of it as three smaller problems: picking a meat, picking a bread and picking a topping. We already know how many ways each of those smaller decisions can be made. It turns out that the answer to the bigger problem can be solved by doing some simple calculations with the number of choices in the smaller ones.

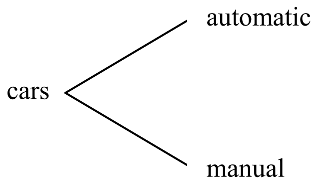
There's one important consideration you have to keep in mind when trying to decompose a large problem into smaller ones. The smaller choices all have to be independent of each other. In other words, your choice of bread can't affect your choice of meat and your choice of meat can't affect your choice of topping. This is obviously the case with our sandwich example but suppose I let you choose two toppings instead of one. Now the problem becomes a little more complicated. If you're allowed to choose the same topping twice then this is the same thing as the original question only you have four smaller choices to make: a bread, a meat, a first topping and a second topping. On the other hand, if I say you can't pick the same topping twice then the topping choices are no longer independent. Why? Because the list of options for the second topping is the same as the list for the first but with your first choice taken out. In other words, I don't know what options you have for the second topping choice until you've made the first one.

Before we try to tackle the deli question let's look at a smaller problem and a convenient way to list out all the possible results. (I know – I said we weren't going to use lists. Just bear with me for a minute. Seeing how I make my list is going to show us how to come up with a much simpler approach to handle bigger problems.)

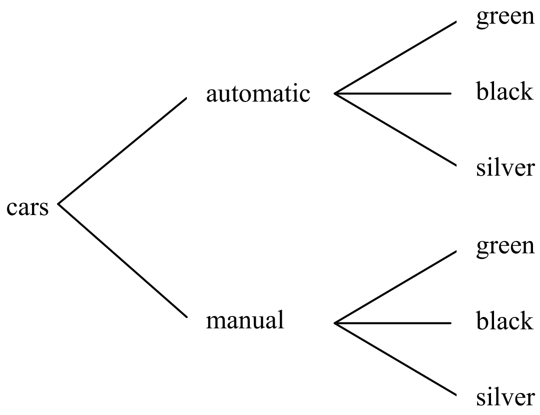
Suppose you're buying a car and you have several options:

Transmission type:	manual or automatic
Color:	green, black or silver

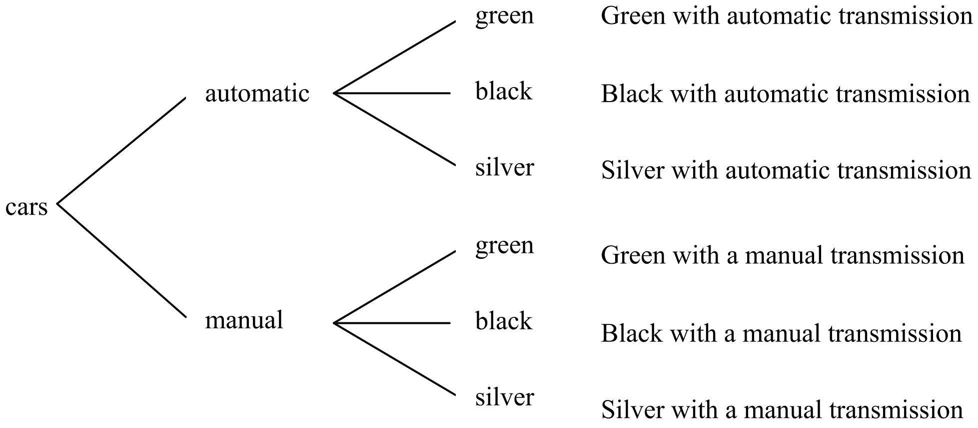
I'd like to figure out how many different types of cars you have to choose from. We can do this by making what's called a "tree diagram". To do this, you start by making a "branch" for each type of transmission.



Now for every transmission branch, I'm going to add a branch for each of the three possible colors.



This set of branches is called a **tree diagram**. If we follow each branch out to the end and write out the choice for every option in the branch, we'll get a list of all the possible car types.



The list of all the possible combinations is called the **sample space.** You can think of it as being a list of all the possible outcomes. A quick count of the list tells you that there are a total of six different cars to choose from.

Notice how we started off with two options (the transmission types) and for each of those we had three color options. When we added the number of colors to the list, it tripled the number of options. This brings us to a general rule that we can use to answer bigger questions like the one about the deli sandwiches.

The Multiplication Principle	Assume that an event E can be decomposed into k events, $E_1, ..., E_k$. If n_1 is the number of ways that event E_1 can occur, n_2 is the number of ways that event E_2 can occur, ..., n_k is the number of ways that event E_k can occur. Then the number of ways that E, the main event, can occur is $n_1 \cdot n_2 \cdot . . . \cdot n_k$.
English Translation	That probably looks like a big mess. Don't let it scare you. The idea is really simple: If you have a list of options the total number of combinations is just the number of options that you have in each category multiplied together.

Looking at our car example, we can apply the Multiplication Principle this way:

$$\text{(total combinations)} = \text{(number of transmission options)} \cdot \text{(number of color options)}$$

$$\text{(total combinations)} = 2 \cdot 3 = 6$$

Which is the same number we got from our tree diagram.

Example 1 – Seating Arrangements

Suppose you have six people who need to be arranged in a row of six seats. How many arrangements are there?

This is another application of the Multiplication Principle but with a little twist. To do these kinds of questions, I like to start by laying out all the spots that I have to fill.

Seat 1	Seat 2	Seat 3	Seat 4	Seat 5	Seat 6

Now, I can fill in the number of choices, i.e. people, that I have for each seat then multiply all those numbers together to get the total. I can put any of the six people in the first seat so that gives me

6					
Seat 1	Seat 2	Seat 3	Seat 4	Seat 5	Seat 6

Now, it gets a little tricky. One of the people has already been seated which means I've only got 5 options for the second seat.

6	5				
Seat 1	Seat 2	Seat 3	Seat 4	Seat 5	Seat 6

From here, the pattern continues for the rest of the spots. I've got 4 people left for the third seat, 3 for the fourth seat, etc. That makes my final table:

6	5	4	3	2	1
Seat 1	Seat 2	Seat 3	Seat 4	Seat 5	Seat 6

If I multiply across the row of numbers, I get $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$ total arrangements.

That situation where we have a descending list of numbers multiplied together comes up often enough that there's a special notation for it called a **factorial**. When we write $n!$ (it's read "n factorial), that means to multiply together all of the integers from n down to 1. For example

$$1! = 1$$

$$2! = 2 \cdot 1 = 2$$

$$3! = 3 \cdot 2 \cdot 1 = 6$$

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

…

Two Special Rules

1. You can only take the factorial of a positive integer, i.e. (-2)! and 1.5! don't have any meaning.
2. 0! = 1 That may seem a little odd but it's mostly for convenience. We'll see some calculations later on where we need 0! to have a value so it's just arbitrarily set equal to 1.

Example 2 – Seating Arrangements

Suppose you have six people who need to be arranged in a row of four seats. How many arrangements are there?

This question is identical to the previous one except I reduced the number of seats. We can set this up exactly the same way as the previous question. First, I'll draw the available seats.

Seat 1	Seat 2	Seat 3	Seat 4

Then fill them in with the number of people available for each seat.

6	5	4	3
Seat 1	Seat 2	Seat 3	Seat 4

If I multiply across the row of numbers, I get 6 · 5 · 4 · 3 = 360 total arrangements.

This is another situation that comes up often enough that there's a special way that we write it.

Permutations

An arrangement of a group of objects is called a **permutation**. The number of permutations of permutations of n items is $n!$. The number of permutations of n items into a group of m is

$${}_nP_m = \frac{n!}{(n-m)!}$$

That's going to be a lot easier to follow with some examples. Let's look back at Example 1. That question, rephrased in terms of permutations, asked, "How many permutations are there of a group of 6 people?". The first part of our definition says that that's just 6! or 6 · 5 · 4 · 3 · 2 · 1 = 720 which is exactly what we calculated.

Example 2, again rephrased using permutations, said, "How many permutations are there of six people in groups of four?" That's the second part of our definition. n is the number of people, six in our example, and m is the number of seats their being grouped into, four in our example. The formula tells us that

$$\text{the number of arrangements} = {}_6P_4 = \frac{6!}{(6-4)!} = \frac{6!}{2!} = \frac{720}{2} = 360$$

Which is also the same number that we calculated.

Example 3 – Committees

The local city council has created a committee with ten people. They want to randomly choose one of the ten to be the chairman and a second to be the secretary. How many combinations do they have to choose from?

Thinking in terms of our permutation formula, this question is asking, "How many groups of two can be made from ten people?" or "How many permutations of ten people are there in groups of two?" Referring to our formula, this means we have $n = 10$ and $m = 2$ so the number of arrangements is

$${}_{10}P_2 = \frac{10!}{(10-2)!} = \frac{10!}{8!} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = 10\cdot 9 = 90$$

Notice how I handled the multiplication. Multiplying out 10! and 8! takes a lot of work and gives you some pretty big numbers. When you've got one factorial divided by another, it's often going to be easier to write them out, reduce the result and then multiply.

In the remaining examples, we'll look at a more general way to apply the Multiplication Principle that, in the end, will allow us to answer probability questions like the one at the beginning of the section.

Example 4 – Counting Sandwiches

Your local deli offers three different kinds of bread, six kinds of meat and four kinds of cheese. How many different sandwiches can you make using those toppings?

We can calculate this using the Multiplication Principle just like we did with the cars.

(total sandwiches) = (number of breads) · (number of meats) · (number of cheeses) (total sandwiches) = 3 · 6 · 4 = 72

That was a lot simpler than trying to list out 72 kinds of sandwiches, hoping that we didn't accidentally miss one.

Example 5 – Counting Dice Results

How many results are there from rolling two dice?

I think this one is easiest to answer using the "blanks" approach:

$$\underset{\text{First Die}}{\underline{\quad 6 \quad}} \qquad \underset{\text{Second Die}}{\underline{\quad 6 \quad}}$$

Since there are 6 options for each of the two dice, the total number of combinations will be 6 · 6 = 36.

Now we're in a position to answer some probability questions, including the one from the beginning of the section.

Example 6 – Probabilities

What's the probability of getting a seven on two dice?

In the previous example, we showed that the total number of outcomes is going to be 36. To get the total number of ways to roll a seven, we'll have to make a list:

$$\{1, 6\} \quad \{2, 5\} \quad \{3, 4\} \quad \{4, 3\} \quad \{5, 2\} \quad \{6, 1\}$$

Since there are six ways to roll a seven that makes our final answer

$$P(\text{rolling a seven}) = \frac{\text{number of ways to get 7}}{\text{total possible results}} = \frac{6}{36} = \frac{1}{6}$$

Example 7 – Probabilities

If you flip a coin three times, what's the probability of getting heads on all three flips?

The total possible combinations of flips will be

$$\underset{\text{Flip 1}}{\underline{\quad 2 \quad}} \qquad \underset{\text{Flip 2}}{\underline{\quad 2 \quad}} \qquad \underset{\text{Flip 3}}{\underline{\quad 2 \quad}}$$

That makes the total number of possible combinations 2 · 2 · 2 = 8. Since only one of those combinations has three heads, our probability must be

$$P(\text{getting 3 heads}) = \frac{\text{number of ways to get 3 heads}}{\text{total possible results}} = \frac{1}{8}$$

Example 8 – Probabilities

Find the probability of getting four or more by rolling two dice.

Trying to write out all of the combinations that give four or more would be a little tedious – there are a lot of them and there's a decent chance that I would miss one. Instead, I'm going to use the compliment rule from the previous section:

$$P(\text{four or more}) + P(\text{three or less}) = 1$$

Calculating P(three or less) is a lot simpler. There are only three combinations that give numbers in that range:

$$\{1, 1\} \quad \{1, 2\} \quad \{2, 1\}$$

That makes

$$P(\text{3 or less}) = \frac{\text{number of ways to get 3 or less}}{\text{total possible results}} = \frac{3}{36} = \frac{1}{12}$$

Substituting that into our compliment formula gives us

$$P(\text{four or more}) + P(\text{three or less}) = 1$$

$$P(\text{four or more}) + \frac{1}{12} = 1$$

$$P(\text{four or more}) = 1 - \frac{1}{12} = \frac{11}{12}$$

The last example is the simple calculation that I promised you when we talked about the Compliment Rule. Instead of trying to count the number of combinations that gave us four or more the rule let us get the answer with a much simpler count, i.e. the combinations that were three or less.

Exercises

1. Write out the sample space of throwing two dice. Confirm that it has the 36 elements that Example 6 predicted it would have.
2. A computer store offers a special on computers with a choice of three different memory options, two different monitors and two different software packages. Draw a tree diagram for the sample space.
3. A couple has three children. Draw a tree diagram that represents the possible combinations of boys and girls.
4. Write a tree diagram that represents the sample space of flipping a coin four times.

Calculate the following group sizes.

5. A state license plate consists of six characters. The first three characters have to be letters and the second three characters have to be numbers, e.g. ABC 123. If both the letters and numbers can be used more than once, how many different possible license plates are there?
6. A computers IP address[2] consists of four numbers between 0 and 127, e.g. 127.0.0.1. How many possible IP addresses are there?
7. A U.S. Social Security Number consists of a group of three numbers, a group of two numbers and a group of four numbers, e.g. 123-00-5432[3]. How many different Social Security Numbers are there?
8. How many possible combinations are there of three dice?
9. If you roll a single die and pick a single card from a deck, how many possible result combinations are there.
10. How many different ways can the seven letters in the word “permute” be arranged? How many words, including words of less than seven letters, can be made from the letters in “permute”?
11. A computer store offers a special on computers with a choice of three different memory options, two different monitors and two different software packages. Use counting methods to confirm that this situation has the sample size that you found in the tree diagram in question (2).
12. Four boys and five girls have to be seated in a row of nine chairs. How many arrangements are there of the kids if a girl has to be in the first seat?
13. Five boys and five girls have to be seated in a row of ten chairs. Find the possible number of arrangements if the first seat has to have a boy and the seats have to alternate between boys and girls.

Calculate the following probabilities.

14. Calculate the probability that a family with two children will have exactly two girls. (Hint: Use the tree diagram from question three.)
15. Calculate the probability that a family with two children will have exactly two girls.
16. Calculate the probability of getting an eleven on two dice.
17. Suppose you’ve been dealt an eight, a nine, a jack and a queen. What’s the probability that the next card will be a ten? (I.e., what’s the probability of drawing to an inside straight?)
18. Calculate the probability that the sum of two dice rolls is less than or equal to four.
19. Calculate the probability that a family with four children has exactly four boys. Calculate the probability that the family has at least one girl.
20. Since 1973, U.S. Social Security numbers have been issued by a central office. The first three digits of a number are assigned based on the state that the application came from. For example, all Social Security numbers assigned to

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[2] An IP address it like the street address of a computer. It’s how other computers, e.g. web sites, know where to send information, e.g. web pages, to it.

[3] Technically, none of the groups can be all 0’s but including that in the calculation makes it significantly more difficult.

applications from New Hampshire start with 001 – 003. Find the probability that a randomly selected Social Security card comes from New Hampshire.

Research Questions

21. Research IP v6, i.e. version 6 of the IP address system. Explain why it's important and how it relates to question (2) above concerning the number of available IP addresses.

Section 1.3 – Addition Rules

Suppose you roll two, standard six sided dice. How likely are you to get a seven or an eleven?

Two Types of Events

To answer that question, there are two situations we have to consider:

1. You roll a seven.
2. You roll an eleven.

Now think about this situation: What's the probability of drawing a six or a heart from a deck of cards. This situation is a little more complicated. We have three possibilities here:

1. You draw a six.
2. You draw a heart.
3. You draw the six of hearts.

The third option, where the other two overlap, is going to be an important part of our calculations. This brings us to our next definition.

Mutually Exclusive	Two events are **mutually exclusive** if they can't occur at the same time.

Example 1 – Mutually Exclusive Events

Are rolling a seven and rolling an eleven mutually exclusive events?

To answer this kind of question, you should ask yourself, "Is it possible for (the first event) and (the second event) to happen at the same time?" In this situation, there's no way to roll an eleven and a seven at the same time so these events would be mutually exclusive.

Example 2 – Mutually Exclusive Events

Are selecting a woman and selecting a Republican from a group of people mutually exclusive events?

It's possible for the person that you pick to be both a woman *and* a Republican so these events are *not* mutually exclusive.

The Probability of Two Mutually Exclusive Events

My original question, "What's the probability of rolling a seven or eleven?", fits this category. This is also our first example that directly relates to a real world situation. In craps, seven and eleven are referred to as "naturals". If you get either of those numbers on your first roll, you automatically win. That makes this value the probability of winning on the first roll of a craps game. We can answer this using our standard formula from Section 1.1:

$$P(7 \text{ or } 11) = \frac{\text{number of sevens} + \text{number of elevens}}{\text{total number of rolls}} = \frac{6+2}{36} = \frac{8}{36} = \frac{2}{9}$$

If we look at that a little more closely, we can get a general rule for calculating the probability of one event *or* another.

$$P(7 \text{ or } 11) = \frac{\text{number of sevens} + \text{number of elevens}}{\text{total number of rolls}} = \frac{\text{number of sevens}}{\text{total number of rolls}} + \frac{\text{number of elevens}}{\text{total number of rolls}} = \frac{6}{36} + \frac{2}{36} = \frac{8}{36} = \frac{2}{9}$$

But take a look at the middle part of that. (number of sevens) / (total number of rolls) is $P(7)$ and (number of elevens) / (total number of rolls) = $P(11)$. Using that, we can rewrite or equation as

$$P(7 \text{ or } 11) = P(7) + P(11)$$

And that gives us a general formula that we can use for mutually exclusive events:

The Probability of Mutually Exclusive Events	If A and B are mutually exclusive events then $P(A \text{ or } B) = P(A) + P(B)$ This can be extended to more than two events in the obvious way, e.g. $P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C)$

Example 3 – Mutually Exclusive Events

A veterinarian has three cats, five dogs and two snakes in the boarding area of his clinic. If an animal is selected at random, find the probability that it's a dog or a cat.

What we're looking for here is $P(\text{dog or cat})$ which we can find using or mutually exclusive probability formula:

$$P(\text{dog or cat}) = P(\text{dog}) + P(\text{cat})$$

Since there are a total of 10 animals and five of them are dogs, we have $P(\text{dog}) = 5/10$. Since three of the animals are cats, we have $P(\text{cat}) = 3/10$. That makes the total probability:

$$P(\text{dog or cat}) = P(\text{dog}) + P(\text{cat}) = \frac{5}{10} + \frac{3}{10} = \frac{8}{10} = \frac{4}{5}$$

Example 4 – Mutually Exclusive Events

Find the probability of rolling 2, 3 or 12 on two dice.

This is the other side of the "natural" situation that we saw at the beginning of the section. Getting 2, 3 or 12 on the first roll in a craps game is an automatic loss. We can find the probability using our mutually exclusive events formula:

$$P(2 \text{ or } 3 \text{ or } 12) = P(2) + P(3) + P(12)$$

There's one way to get a 2, i.e. $\{1, 1\}$, two ways to get a 3, i.e. $\{1, 2\}$ and $\{2, 1\}$, and one way to get a 12, i.e. $\{6, 6\}$ which makes the probability:

$$P(2 \text{ or } 3 \text{ or } 12) = P(2) + P(3) + P(12) = \frac{1}{36} + \frac{2}{36} + \frac{1}{36} = \frac{4}{36} = \frac{1}{9}$$

The Probability of Two Non-Mutually Exclusive Events

At this point, you may be thinking that the mutually exclusive probability formula is a pretty trivial thing and you would be right. We could have figured out all of the probabilities in the first two examples using the formula from Section 1.1. Where things get a little more interesting is with the non-mutually exclusive case. Think about the situation that I mentioned at the beginning of the section: Find the probability of drawing a six or a heart. If we tried to apply the mutually exclusive formula to this, we'd get:

$$P(\text{heart or six}) = P(\text{heart}) + P(\text{six})$$

But there's a problem here. If we did that, one card, the six of hearts, would get counted twice, once in the $P(\text{heart})$ calculation and again in the $P(\text{six})$ calculation. To correct for that, we have to subtract back out all of the "overlapping events", i.e. the events that occur in both categories. That gives us our next formula:

The Probability of Non-Mutually Exclusive	If A and B are not mutually exclusive events then $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

Before we go any further let me make something clear: You *cannot* extend this to three events the way you could with the mutually exclusive case, i.e. you can't say that $P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C) - P(A \text{ and } B \text{ and } C)$. There is a way to extend the formula but it's a more complex situation than we'll be discussing.

Example 5 – Non-Mutually Exclusive Events

Find the probability of drawing a six or a heart from a deck of cards.

What we're looking for here is *P*(six or heart). Since these events overlap, i.e. it's possible to draw a card that's a six and a heart, we have to use the non-mutually exclusive formula:

$$P(\text{six or heart}) = P(\text{six}) + P(\text{heart}) - P(\text{six and heart})$$

Of the 52 cards in the deck, 4 are sixes so *P*(six) = 4/52. Twelve of the 52 cards are hearts so *P*(heart) = 12/52. Finally, there's only one card that's a six and a heart so *P*(six and heart) = 1/52. If we substitute those values into our formula, we get

$$P(\text{six or heart}) = P(\text{six}) + P(\text{heart}) - P(\text{six and heart})$$

$$P(\text{six or heart}) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

Example 6 – Non-Mutually Exclusive Events

Rolling two even numbers or numbers that total six on two dice.

The combinations that add up to six are

$$\{1, 5\} \quad \{2, 4\} \quad \{3, 3\} \quad \{4, 2\} \quad \{5, 1\}$$

Two of those combinations also have two even numbers, {2, 4} and {4, 2}, so this is as non-mutually exclusive situation.

We can apply our counting methods to find the number of pairs that have two even numbers. There are two dice and each of them has three even numbers so the total "even pairs" would be

$$3 \cdot 3 = 9$$

Applying our formula gives us:

$$P(\text{evens or totals six}) = \\ P(\text{evens}) + P(\text{totals six}) - P(\text{evens and totals six})$$

$$P(\text{evens or totals six}) = \frac{9}{36} + \frac{5}{36} - \frac{2}{36} = \frac{12}{36} = \frac{1}{3}$$

You might have noticed that, when I found the probabilities in Example 5, I didn't reduce the fractions. Most of the time, reducing fractions is going to make them easier to work with but, in this case, I knew I was going to have to add/subtract the fractions in the end so, by not reducing them, I kept them all with the same denominator. That meant I didn't have to convert them all back to having a common denominator when it came time to do the final arithmetic.

Example 7 – Non-Mutually Exclusive Events

A survey of 100 people found that 87 of them owned a car, 12 of them owned a motorcycle and 4 of them owned both. Find the probability that a randomly selected person owns a motorcycle or a car but not both.

It may not be immediately obvious to you that this is a non-mutually exclusive case. To see it, think about what our two groups are: people who own cars and people who own motorcycles. The question explicitly tells us that 4 people fall into both categories which is why this is a non-mutually exclusive situation.

Applying out formula gives us:

$$P(\text{car or motorcycle}) = P(\text{car}) + P(\text{motorcycle}) - P(\text{car and motorcycle})$$

$$P(\text{car or motorcycle}) = \frac{87}{100} + \frac{12}{100} - \frac{4}{100} = \frac{95}{100} = \frac{19}{20}$$

Exercises

Classify the following situations as mutually exclusive or non-mutually exclusive.

1. Drawing an even numbered card and drawing an odd numbered card.
2. Rolling an even number and rolling an odd number.
3. Randomly selecting an animal from a veterinary clinic that's a mammal and randomly selecting an animal that's a reptile.
4. Drawing a three or a black card from a standard deck.

5. Randomly selecting a person who was born in New York and randomly selecting a person who was born in Maryland.
6. Selecting a high school student who is a senior and selecting a high school student who is a girl.
7. Randomly selecting an animal from a veterinary clinic that's a mammal and randomly selecting an animal that has four legs.
8. Selecting a book with an even number of pages and selecting a book with an odd number of pages.
9. Selecting a book of poetry and selecting a book written in Spanish.
10. Selecting a type of tree and selecting a plant that has flowers.

Find the following probabilities.

11. The probability of drawing an ace , a seven or a king from a deck of cards.
12. The probability of drawing an ace or a spade from a deck of cards.
13. A researcher pulls fish from a river and finds 35 male lake trout, 30 female lake trout, 12 male silver trout and 16 female silver trout. Find the probability that a randomly selected fish is a lake trout.
14. The probability of drawing a card with an even number or a black card from a deck of cards.
15. A bag of M&M's contains 60 red, 80 orange, 75 blue, 75 green and 85 yellow candies. What's the probability of drawing an M&M that's red, blue or green from the bag?
16. The probability of picking a random state whose name starts with a vowel or that borders the Pacific ocean.
17. A pollster surveys 1000 people and finds that 325 plan to buy a new refrigerator in the next month, 120 plan to buy a new dishwasher and 37 plan to buy both. Find the probability that some plans to buy a new refrigerator or a new dishwasher.
18. The probability of rolling two dice and getting a result whose sum is less than or equal to five or that has a one on at least one of the dice.

Section 1.4 – Multiplication Rules

A raffle sells 500 tickets. If there's one first place ticket and one second place ticket, what's the probability that a person who buys two tickets gets both the first and second place prize?

In the previous section, we talked about a single event with multiple possibilities, e.g. a single roll of a die that gives a three or a five. In this section, we're going to look at situations where you have multiple events that take place in sequence like the drawing of two raffle tickets. Before we do that, we've got two more terms you should be aware of.

Independent vs. Dependent Events	Two events are **dependent** if one event changes the possible outcomes of the other. The events are **independent** if one event occurring doesn't change the outcomes of another.

We can illustrate both of these types with our raffle example.

Dependent Events	Suppose you draw a ticket (the first event) then throw it away and draw the second ticket (the second event). The probability of getting the first ticket will be 1/500, one winning ticket out of 500. When you go to draw the second winning ticket there will only be 499 tickets left in the container. That makes the probability of getting the second ticket only 1/499. These events are dependent because, even though they're both drawing a single ticket, their probabilities are different. Not replacing the ticket caused the second probability to be slightly

	greater than the first.
Independent Events	Suppose you draw a ticket (the first event) then *put it back in the container* and draw the second ticket (the second event). In this case, there are 500 tickets in the container for both draws so they both have the same probability of occurring, 1/500. In this case, the first event didn't affect the probability of the second so the events are independent.

The nice thing about these events is that, unlike the "or" problems in the previous section, we can use the same equation for both cases.

The "and" Probability	$P(A \text{ and } B) = P(A) \cdot P(B)$ This can be extended to more than two events in the obvious way, e.g. $P(A \text{ and } B \text{ and } C) = P(A) \cdot P(B) \cdot P(C)$

Example 1 – Independent Events

If a raffle sells 1000 tickets then picks one first place and one second place ticket, find the probability of winning both prizes if the tickets are replaced after each draw.

As we saw in the previous discussion:

$$P(\text{first prize}) = \frac{1}{500} \quad \text{and} \quad P(\text{second prize}) = \frac{1}{500}$$

Since the number of tickets in the raffle container is the same both times. That makes our final answer

$$P(\text{first and second}) = P(\text{first prize}) \cdot P(\text{second prize})$$

$$P(\text{first and second}) = \frac{1}{500} \cdot \frac{1}{500}$$

$$P(\text{first and second}) = \frac{1}{250000}$$

Example 2 – Dependent Events

If a raffle sells 1000 tickets then picks one first place and one second place ticket, find the probability of winning both prizes if the tickets are *not* replaced after each draw.

As we saw in the previous discussion:

$$P(\text{first prize}) = \frac{1}{500} \quad \text{and} \quad P(\text{second prize}) = \frac{1}{499}$$

Since the number of tickets in the raffle container goes down by one after each draw. That makes our final answer

$$P(\text{first and second}) = P(\text{first prize}) \cdot P(\text{second prize})$$

$$P(\text{first and second}) = \frac{1}{500} \cdot \frac{1}{499}$$

$$P(\text{first and second}) = \frac{1}{249500}$$

Example 3 – Gender of Children

Find the probability that a family with three children is all boys.

First, notice that $P(\text{boy}) = 1/2$ since there are two possible genders, one of which is a boy. Now we can apply our "and" formula to get the final answer.

$$P(\text{three boys}) = P(\text{boy and boy and boy})$$

$$P(\text{three boys}) = P(\text{boy}) \cdot P(\text{boy}) \cdot P(\text{boy})$$

$$P(\text{three boys}) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$$

$$P(\text{three boys}) = \frac{1}{8}$$

Example 4 – Survey Responses

Find the probability that a family with three children has at least one girl.

To apply the "and" formula directly would be a challenge here. We would need to consider the case where the first child is a boy and the second two are girls then the case where the first two children are boys and the third is a girl, etc. There are several cases and it would be easy to miss one. It's easier to apply our complementary events rule:

$$P(\text{no girls}) + P(\text{at least one girl}) = 1$$

But $P(\text{no girls})$ is the same as $P(\text{three boys})$ which we calculated in Example 3. This makes our final answer

$$P(\text{no girls}) + P(\text{at least one girl}) = 1$$

$$P(\text{three boys}) + P(\text{at least one girl}) = 1$$

$$\frac{1}{8} + P(\text{at least one girl}) = 1$$

$$P(\text{at least one girl}) = 1 - \frac{1}{8} = \frac{7}{8}$$

Example 5 – Survey Responses

A pollster found that, out of 100 people, 35 favored a new bond measure, 60 opposed it and 5 had no opinion. If four people are selected at random, what's the probability that at least one has an opinion?

This situation is similar to the previous one. Trying to list out all the cases that have at least one opinion would be too difficult and error prone. It's easier to apply our complementary events rule:

$$P(\text{none have an opinion}) + P(\text{at least one has an opinion}) = 1$$

We can calculate *P*(none have an opinion) using our formula. (Remember that the question specified that three people are chosen.)

$$P(\text{none have an opinion}) = P(\text{no opinion and no opinion and no opinion})$$

$$P(\text{none have an opinion}) = P(\text{no opinion}) \cdot P(\text{no opinion}) \cdot P(\text{no opinion})$$

$$P(\text{none have an opinion}) = \frac{5}{100} \cdot \frac{5}{100} \cdot \frac{5}{100}$$

$$P(\text{none have an opinion}) = \frac{125}{1000000} = \frac{1}{8000}$$

Substituting that into our complementary events rule gives us:

$$P(\text{none have an opinion}) + P(\text{at least one has an opinion}) = 1$$

$$\frac{1}{8000} + P(\text{at least one has an opinion}) = 1$$

$$P(\text{at least one has an opinion}) = 1 - \frac{1}{8000}$$

$$P(\text{at least one has an opinion}) = \frac{8000}{8000} - \frac{1}{8000} = \frac{7999}{8000}$$

Exercises

Calculate the following probabilities.

1. A research lab is testing a new drug and finds that, out of 50 patients, 7 had an undesirable side effect and the rest didn't. Find the probability that, if three patients are chosen, the first as the side effect but the second two don't.
2. The probability of drawing a queen and an ace if the cards are replaced after each draw.
3. The probability of getting sixes on each of three rolls of a die.
4. The probability that a family of four has all girls.
5. The probability that a family of four has at least one girl.
6. The probability of drawing a red card then a red card if the cards are *not* replaced after each draw.
7. A fisherman catches four snapper and one grouper. Find the probability that the next two fish he catches are both grouper.
8. Of 125 people surveyed, 78 planned to take a vacation in the next year. Find the probability that the next two people surveyed don't plan to take a vacation.
9. A church raffle has one first prize and two second prizes. If the church sells 200 tickets and the tickets are drawn without replacement, find the probability of winning one of the two second prizes.

10. The probability of getting even numbers on each of four rolls of a die.